4/24/26

Exam 2 —

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§6.2 Determinants

| | | |
|---|---|---|
| all A's | 90–100 | mean 90.4 |
| all B's | 80–89 | median 92 |
| all C's | 70–79 | |
| D | 60–69 | |
| F | <60 | |

Q: what effect do natural changes on a matrix have on a determinant?

- transpose
- inverse
- elem row operations

Theorem 6.2.1

If $A$ is a square matrix then $\det A = \det A^T$

(skip proof, give idea)

Inverse: we'll come to this

Elem Row Operations (recall)

Th 6.2.3

(a) If $B$ is obtained from $A$ by dividing a row of $A$ by a scalar $k$ then

$$\det B = \left(\frac{1}{k}\right)(\det A)$$

i.e. $\det A = k \cdot \det B$

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det A = 1$ divide R1 by 2

$B = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ $\det B = \frac{1}{2} \cdot \det A$

(b) If $B$ is obtained from $A$ by swapping 2 rows then

$$\det B = -\det A$$

(c) If $B$ is obtained from $A$ by adding a mult. of a row of $A$ to another row then $\det B = \det A$

Th Let $C$ be an $n \times n$ matrix in RREF.

If $C = I_n$ then $\det C = 1$

If $C \neq I_n$ then $\det C = 0$ (row of 0's at the bottom)

Ex $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \quad \det = 1$

$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 0$ (each choice of a prod taking one from each row and one from each col must use row 3, so it's 0)

Algorithm 6.2.5 – Using Gauss-Jordan elim to compute the determinant.

Let $A$ be an $n \times n$ matrix

Row reduce $A$ to get RREF.

If $RREF(A) \neq I_n$ then $\det(A) = 0$. Done.

Assume $RREF(A) = I_n$.

Assume in finding $RREF(A)$ you

- perform type (b) $s$ times
- perform type (a) $r$ times using scalars $k_1, k_2, \dots, k_r$
- we don't care how many times you perform type (c)

Then $\det(A) = (-1)^s k_1 k_2 \cdots k_r \det B$ (use blue version above)

Actually there's a shortcut: if you row reduce to a matrix for which you know the determinant, you can stop

Ex Find det(A) where
$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 9 & 13 \\ 2 & 4 & 3 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 9 & 13 \\ 2 & 4 & 3 & 5 \end{bmatrix} \xrightarrow[\text{R1 by 2}]{\text{type (a) divide}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 6 & 9 & 13 \\ 2 & 4 & 3 & 5 \end{bmatrix} \xrightarrow{\text{type (c)}} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$$\xrightarrow[\text{R3, R4}]{\text{type (b) swap}} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

this matrix has
det = 3

We did $s=1$ swap

We did type (a) once: divide by 2

We did some type (c), but we don't care

$$\Rightarrow \det A = (2)(-1)^1(3) = -6$$

Other important properties of the determinant:

Th 6.2.6

If $A, B$ are $n \times n$ matrices and $m$ is a positive integer then

$$\det(AB) = \det(A) \cdot \det(B)$$

$$\det(A^m) = \left[\det(A)\right]^m$$

Ex
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ 9 & -2 \end{bmatrix}$$

det $\quad -2 \quad\quad -4 \quad\quad 8 \quad$ ✓

Th

If $A$ is similar to $B$ then $AS = SB$ for some invertible matrix $S$, so $(\det A)(\det S) = (\det S)(\det B)$

$$\Rightarrow \det A = \det B$$

Ie similar matrices have the same determinant

How about inverses?

If $A$ is invertible then $AA^{-1} = I_n$

So $\det(A) \cdot \det(A^{-1}) = \det(I_n) = 1$

So we've shown

theorem 6.2.8

$$\det(A^{-1}) = \frac{1}{\det(A)}$$